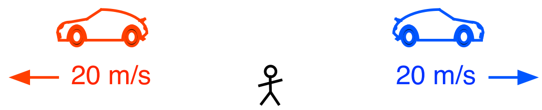
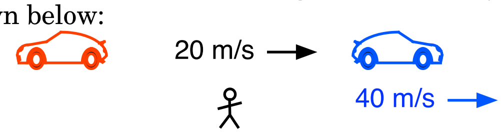
Relative Velocity

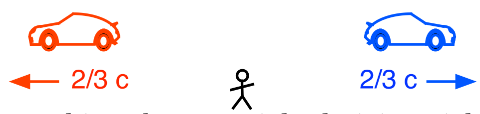
Spacetime diagrams are also useful to see how different reference frames will measure different speeds for an object. First, consider a classic question by imagining that you see two cars going in opposite directions with the same speed (let's say 20 m/s) as shown below:



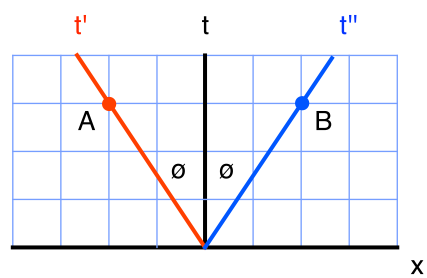
How fast would the cars be moving relative to each other? Hopefully it makes sense that each car would measure the other car to be moving at 40 m/s away from it. The left (red) car frame of reference is shown below:



This is traditional "Galilean" relativity, and the way we looked at things way back at the start of the year in relative motion. The speeds simply add up. When the speeds approach the speed of light however, it is more complicated. Now imagine we have really fast cars traveling away from us close to the speed of light, shown below:



Someone who did not know anything about special relativity might say that the speed of the cars relative to each other is 2/3 c + 2/3 c = 4/3 c, but that can't be correct because that is faster than light. We know the correct answer has to be something less than the speed of light – but what is it?

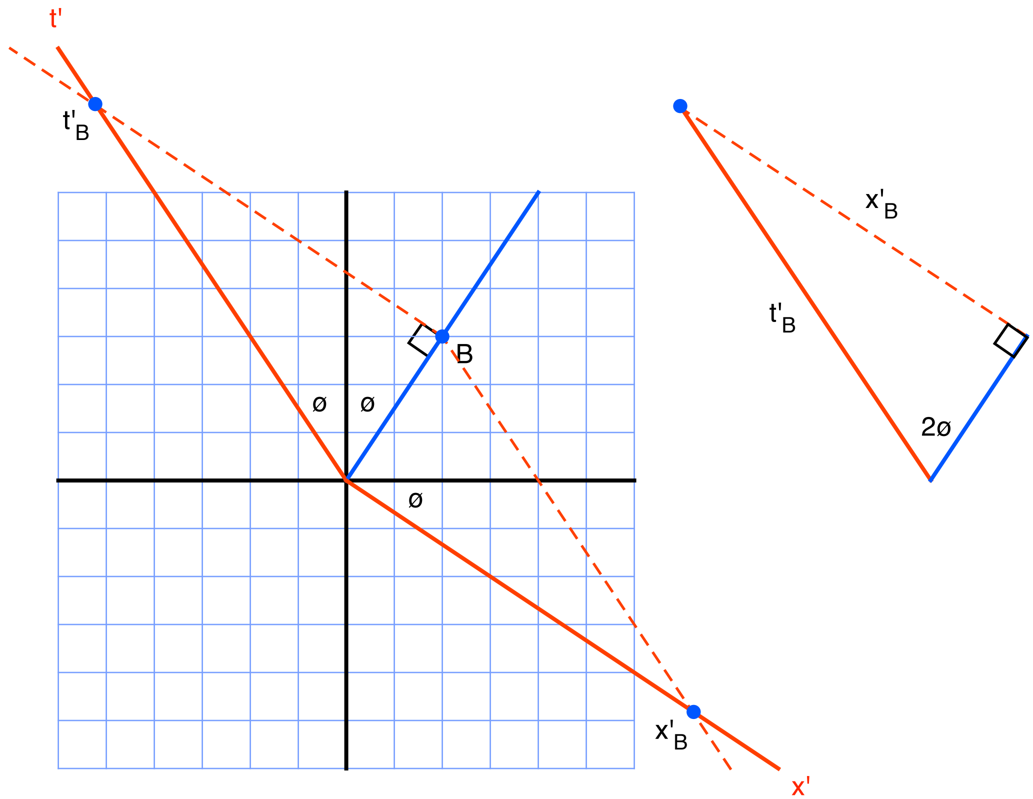


We will figure this out with a spacetime diagram. The diagram to the right shows the situation in your reference frame. The worldlines of the red and blue cars are shown in red and blue. To simplify things, you and the two cars all start at the origin. The events labeled A and B in the diagram are just the locations of the cars at some later time in your reference frame. The actual numbers don't matter, but notice in the diagram that the time is 3 and the locations of the cars are –2 and 2. Therfore, the velocity of the red car in the S frame (your frame, also called the rest frame) is $v_A = x_A/t_A = -2/3$ and the velocity of the blue car is $v_B = x_B/t_B = 2/3$. Those are the velocities of the cars with respect to you. To find the velocities of the cars with respect to each other, we need to do the same thing: in S' (red car) the velocity of the blue car is $v'_B = x'_B/t'_B$ and in S" (blue car) the velocity of the red car is $v'_A = x'_A/t'_A$.

Remember that in a spacetime diagram, v = tan ø.

Let's do the S' reference frame here. First, we will have to draw in the x' axis. Then to find the coordinates of B in S', we have to draw in two lines passing through event B: one parallel to x' to find t'_B and one parallel to t' to find x'_B. This is shown below. (We need a little more room on the graph, so it will be bigger.)

Relative Velocity



Notice how we made a parallelogram, since we have two sets of parallel lines. Even better, notice how the parallelogram is made of two right triangles (only one right angle symbol is shown.) To make it really clear what we have, the top left triangle is shown again to the right.

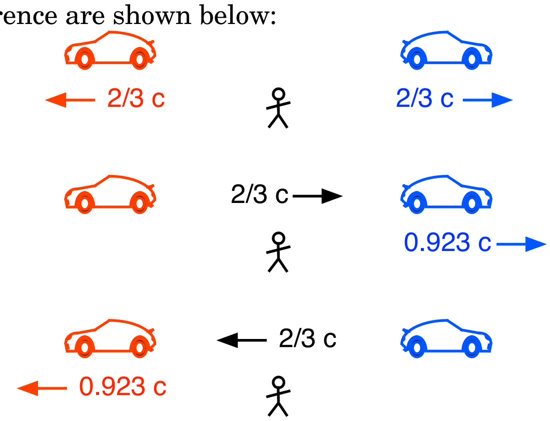
Therefore, we can say

$$\boxed{v'_B = \frac{x'_B}{t'_B} = \sin 2ø \quad \text{where} \quad \tan ø = \frac{x_B}{t_B} = v_B}$$

Back to our original question:

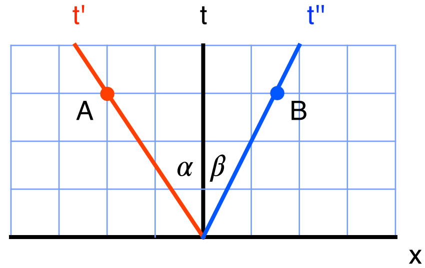
$$ø = tan^{-1}(\tfrac{2}{3}) = 33.69^{o} \qquad \text{so} \qquad v'_B = \sin(2 \cdot 33.69) = 0.923$$

All three frames of reference are shown below:

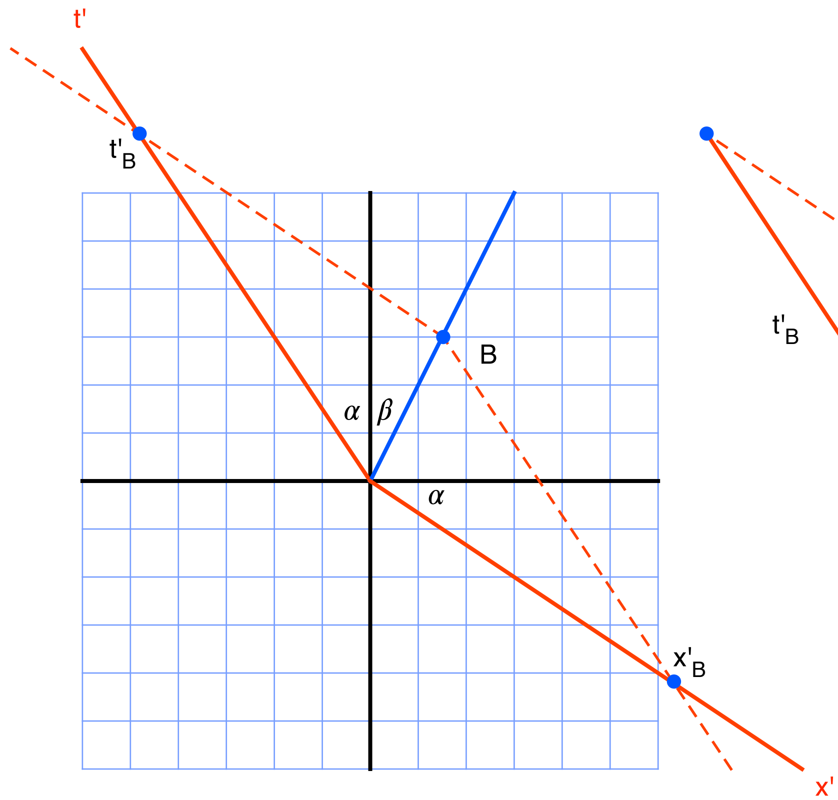


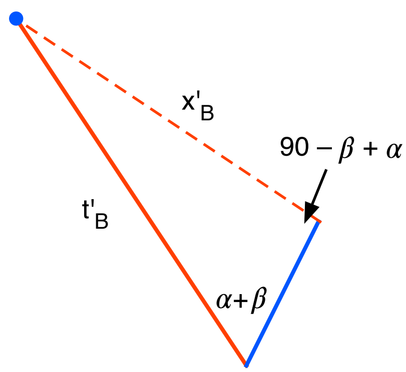
Relative Velocity

If the two cars have different speeds in your reference, the situation is only slightly more complicated. The spacetime diagram for this is shown below:



Again, the cars both start at the origin, but this time the slopes of the worldlines for the cars are different. We go through the same process to find the coordinates of B in the S' frame, shown below left:





It looks very similar to the previous situation. We still have a parallelogram, but this time there are no right angles. As before, the top left triangle is drawn above right to make the triangle clear. This time, we will have to use the Law of Sines as follows:

$$\frac{x'_B}{\sin(\alpha+\beta)} = \frac{t'_B}{\sin(90-\beta+\alpha)}$$

Therefore, we can say

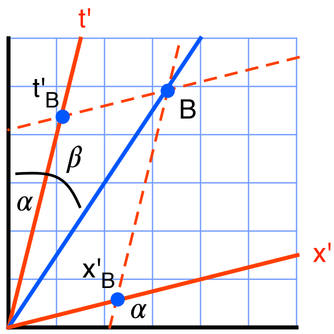
$$\boxed{v'_B = \frac{x'_B}{t'_B} = \frac{\sin(\alpha+\beta)}{\sin(90-\beta+\alpha)} = \frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} \qquad \text{where} \qquad \tan\alpha = v_A \text{ and } \tan\beta = v_B}$$

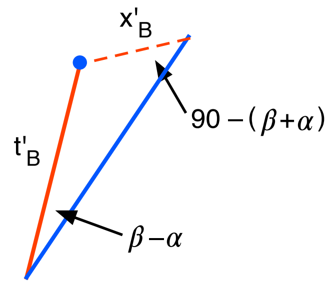
Notice that it does reduce to our previous answer if the two speeds are the same.

Relative Velocity

What if the two cars were approaching you instead of going away? The answer is the same. If we extend all the worldlines to the negatives, that would show the two cars traveling towards you, all meeting at the origin, and then moving away from you. The S' axes would be the exact same and finding the velocity of B in the S' frame really is just finding the slope of the blue line in the S' frame, so nothing really changes.

Lastly, what if both the cars were moving to the right of the observer? The equation would be the same – just make one of the angles negative. I suppose it would be better to do the derivation for that situation, and then make one of the velocities negative when they go in opposite directions. That spacetime diagram is a little cramped, but you still get a parallelogram and then use the Law of Sines on one of the triangles that make up the parallelogram. Ultimately, I think going in opposite directions is the more interesting situation, so that's why I did it that way, but the spacetime diagram is shown below if you would like to try it:





Using some numbers from the very first set of problems, what if the two cars had speeds of 0.5c and 0.8c according to you. What is the relative speed of the two cars with respect to each other?

First, find the slopes of the two cars' worldlines:

$$\tan\alpha = 0.5 \ \& \ \tan\beta = 0.8$$

So that

$$\alpha = 26.57 \ \& \ \beta = 38.66$$

So that finally

$$v'_B = \frac{\sin(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{\sin(26.57+38.66)}{\cos(26.57-38.66)} = 0.929\ c$$

All three reference frames are shown in the diagram below